Theorem $f, g : [a,b] \to \mathbb{R}$ integrable, $c \in \mathbb{R}$

$\Rightarrow$ (a) cf is integrable

(b) $f+g$ is integrable

Proof (a) ✓ last class

(b) showed last time

(1) $$\boxed{\begin{aligned} U(f+g, P) &\le U(f,P) + U(g,P) \quad \text{for any partition } P \\ L(f+g, P) &\ge L(f,P) + L(g,P) \quad \text{" " "} \end{aligned}}$$

(boils down to $\sup(x+y,\ x \in X, y \in Y) \le \sup\{x \in X\} + \sup\{y \in Y\}$

$\inf\{x+y, \ldots \ge \inf \text{"} + \inf \text{"}$)

pick $\varepsilon > 0$

$$\boxed{\begin{array}{l} f \text{ integrable} \Rightarrow \exists \text{ partition } P_1 \text{ s.t. } U(f,P_1) - L(f,P_1) < \varepsilon/2 \\ g \quad '' \quad '' \quad '' \quad '' \quad P_2 \text{ s.t. } U(f,P_2) - L(f,P_2) < \varepsilon/2 \end{array}} \quad (2)$$

Let $P = P_1 \cup P_2$

showed last time:

$$\boxed{\begin{array}{ll} U(f,P) \leq U(f,P_1) & U(g,P) \leq U(g,P_2) \\ L(f,P) \geq L(f,P_1) & L(g,P) \geq L(g,P_2) \end{array}} \quad (3)$$

$$\Rightarrow U(f+g,P) - L(f+g,P) \overset{(1)}{\leq} U(f,P) + U(g,P) - L(f,P) - L(g,P)$$

$$\overset{(3)}{\leq} \underbrace{U(f,P_1) - L(f,P_1)}_{< \varepsilon/2} + \underbrace{U(g,P_2) - L(g,P_2)}_{< \varepsilon/2 \text{ by } (2)}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

__Theorem__ (a) $f, g : [a,b] \to \mathbb{R}$ integrable, $f(x) \leq g(x)$ $\forall x \in [a,b]$

$$\Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

(b) g cont., $g(x) \geq 0$ for all $x \in [a,b]$

$$\int_a^b g(x)\,dx = 0 \Rightarrow g(x) = 0 \text{ for all } x \in [a,b]$$

(Remark: not true if g is not continuous)

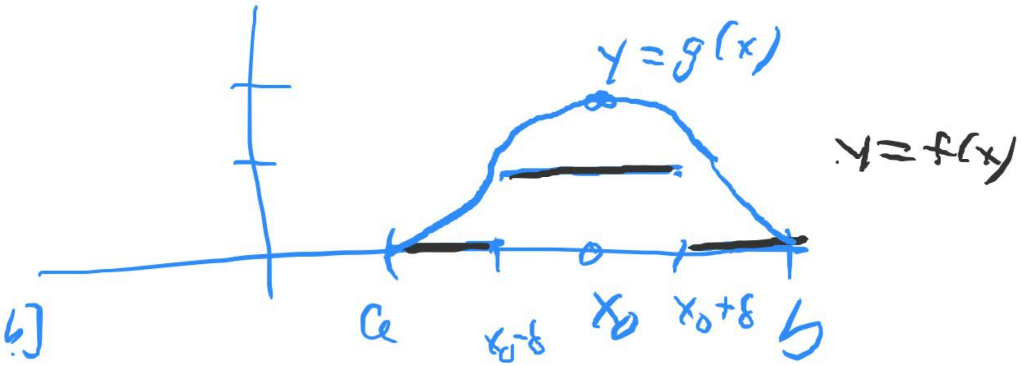
Proof. (a) Let $h(x) = g(x) - f(x)$. enough to show: $\boxed{\int_a^b h(x)\,dx \geq 0}$

(indeed: $\int_a^b g(x)\,dx = \int_a^b \underbrace{g(x) - f(x)}_{h(x)} + f(x)\,dx$

$$\underset{\text{Theorem (b)}}{=} \underbrace{\int_a^b h(x)\,dx}_{\geq 0} + \int_a^b f(x)\,dx \quad)$$

$\Rightarrow$ claim!

Let $f(x) = \begin{cases} g(x_0)/2 & \text{if } |x-x_0| \le \delta \\ 0 & \text{if } |x-x_0| > \delta \end{cases}$



<u>check:</u> f is integrable!

by def. $g(x) \ge f(x)$ for all $x \in [a,b]$

$$\Rightarrow \int_a^b g(x)\,dx \underset{a}{\ge} \int_a^b f(x)\,dx = \frac{g(x_0)}{2}\left(x_0+\delta - (x_0-\delta)\right)$$

$$= \frac{g(x_0)}{2} \cdot 2\delta = g(x_0)\,\delta > 0$$

$\Rightarrow$ claim.

Theorem f integrable on $[a,b]$ $\Rightarrow$ $|f|$ integrable on $[a,b]$

$\left(\quad |f|(x) = |f(x)| \quad \right)$

and $\boxed{\int_a^b |f|\,dx \geq \left|\int_a^b f(x)\,dx\right|}$

Proof. main point: show that $|f|$ is integrable!

Δ-inequality $\Rightarrow$ $|c| - |d| \leq |c-d|$ for any $c, d \in \mathbb{R}$

$\Rightarrow$ $|f(x_n)| - |f(y_n)| \leq |f(x_n) - f(y_n)|$ for any $x_n, y_n \in \mathbb{R}$

let I be any interval

(x_n) sequence s.t. $|f(x_n)| \to M(|f|, I)$

(y_n) " " $|f(y_n)| \to m(|f|, I)$

$$\Rightarrow \quad M(|f|, I) - m(|f|, I) = \lim_{n\to\infty} \underbrace{|f(x_n)| - |f(y_n)|}_{\leq\ |f(x_n) - f(y_n)|}$$

$$\leq \sup |f(x_n) - f(y_n)|$$

$$\leq M(f, I) - m(f, I)$$

$\Rightarrow \quad U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$

for any partition P

pick $\varepsilon > 0$

f integrable $\Rightarrow \exists P$ s.t. $U(f, P) - L(f, P) < \varepsilon$

by last inequality $\Rightarrow \quad U(|f|, P) - L(|f|, P) < \varepsilon$

$\Rightarrow |f|$ integrable.

to prove inequality observe

$$-|f(x)| \leq f(x) \leq |f(x)| \quad \text{for all } x \in [a, b]$$

$$\underset{\text{Th.}}{\Rightarrow} \quad -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$$\Rightarrow \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx \quad \checkmark$$

Theorem (IMV theorem for integrals)

$f : [a,b] \to \mathbb{R}$ continuous

$\Rightarrow \exists$ at least one $y \in [a,b]$ s.t.

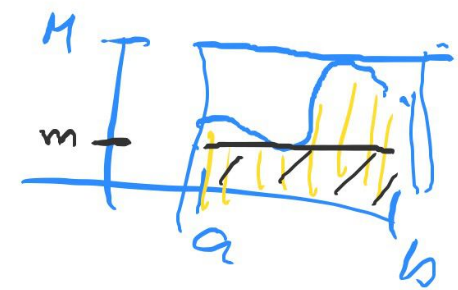
$$f(y) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Proof. f cont. $\Rightarrow \exists\ x_0$ s.t. $m = f(x_0) \leq f(x) \quad \forall x \in [a,b]$

$\exists\ y_0$ s.t. $M = f(y_0) \geq f(x) \quad$ " . . .

$$\Rightarrow \quad m \leq f(x) \leq M \qquad \forall x$$

$$\Rightarrow \quad \underbrace{m(b-a)}_{=\int_a^b m\,dx} \leq \int_a^b f(x)\,dx \leq M(b-a)$$



$$\Rightarrow \quad \boxed{m \leq \frac{1}{(b-a)} \int f(x)\,dx \leq M.}$$

by IMV for cont. functions

$$\Rightarrow \exists\, y \text{ s.t.}$$

$$f(y) = \underbrace{\frac{1}{(b-a)} \int_a^b f(x)\,dx}_{\text{is between } m = f(x_0) \text{ and } M = f(y_0)}$$

As $h(x) \geq 0$ for all x $\Rightarrow$ $L(h,P) \geq 0$ for any partition P

$$\Rightarrow \quad L(h) = \sup_{P \text{ a partition}} L(h,P) \geq 0$$

$$\| $$

$$\int_a^b h(x)\,dx$$

($h(x)$ is integrable by Th. (b))

✓

(b) Assume $\exists\, x_0$ s.t. $g(x_0) > 0$

g cont. $\Rightarrow \exists\, \delta > 0$ s.t. $|g(x) - g(x_0)| < \frac{g(x_0)}{2} = \varepsilon$

if $|x - x_0| < \delta$

in particular $g(x) > \frac{g(x_0)}{2}$ if $|x - x_0| < \delta$